Generating Endomorphism Rings of Infinite Direct Sums and Products of Modules

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1 Introduction

In [2] Macpherson and Neumann show that for any infinite set Ω the full permutation group $\mathrm{Sym}(\Omega)$ is not the union of a chain of $\leq |\Omega|$ proper subgroups, while in [1] Bergman shows that if U is a generating set for $\mathrm{Sym}(\Omega)$ as a group, then there exists a positive integer n such that every element of $\mathrm{Sym}(\Omega)$ can be written as a group word of length $\leq n$ in elements of U. We will modify the arguments used in these two papers to prove analogous statements for endomorphism rings of infinite direct sums and products of isomorphic modules. Namely, we will show that if R is a unital associative ring, M is a left R-module, Ω is an infinite set, and N is either $\bigoplus_{i\in\Omega} M$ or $\prod_{i\in\Omega} M$, then the ring $\mathrm{End}_R(N)$ is not the union of a chain of $\leq |\Omega|$ proper subrings, and also that given a generating set U for $\mathrm{End}_R(N)$ as a ring, there exists a positive integer n such that every element of $\mathrm{End}_R(N)$ is represented by a ring word of length at most n in elements of U. (The notion of ring word will be made precise below.)

2 Moieties

First we will fix some notation and prove two somewhat technical lemmas. For the rest of this section R will denote a ring, M a nonzero left R-module, and Ω an infinite set. N will denote either $\bigoplus_{i\in\Omega} M$ or $\prod_{i\in\Omega} M$ (all the arguments will work under either interpretation), and E will denote $\mathrm{End}_R(N)$. Endomorphisms will be written on the right of their arguments. Also, given a subset $\Sigma \subseteq \Omega$, we will write M^Σ for the R-submodule of N corresponding to Σ, so in particular, $N = M^\Sigma \oplus M^{\Omega\setminus\Sigma}$. If $\Sigma \subseteq \Omega$ and $U \subseteq E$, then let $U_{\{\Sigma\}} = \{f \in U : M^\Sigma f \subseteq M^\Sigma,\ M^{\Omega\setminus\Sigma} f \subseteq M^{\Omega\setminus\Sigma}\}$ and $U_{[\Sigma]} = \{f \in U : M^\Sigma f \subseteq M^\Sigma,\ M^{\Omega\setminus\Sigma} f = \{0\}\}$. We will also say that a subset $\Sigma \subseteq \Omega$ is *full* with respect to $U \subseteq E$ if the set of endomorphisms of M^Σ induced by members of $U_{\{\Sigma\}}$ is all of $\mathrm{End}_R(M^\Sigma)$. Finally, $\Sigma \subseteq \Omega$ is called a *moiety* if $|\Sigma| = |\Omega| = |\Omega\setminus\Sigma|$.

We will now prove analogs of Lemmas 3 and 4 of [1].

Lemma 1. *Suppose that a subset $U \subseteq E$ has a full moiety $\Sigma \subseteq \Omega$. Then there exist elements $x, y \in E$ such that $E = yUy + yUyx + xyUy + xyUyx$.*

Proof. Let π_Σ denote the projection from N to M^Σ along $M^{\Omega\setminus\Sigma}$ and $\pi_{\Omega\setminus\Sigma}$ denote the projection from N to $M^{\Omega\setminus\Sigma}$ along M^Σ. Then for any $f \in E$ we can write $f = \pi_\Sigma f \pi_\Sigma + \pi_\Sigma f \pi_{\Omega\setminus\Sigma} + \pi_{\Omega\setminus\Sigma} f \pi_\Sigma + \pi_{\Omega\setminus\Sigma} f \pi_{\Omega\setminus\Sigma}$. We also note that $\pi_\Sigma U \pi_\Sigma = E_{[\Sigma]}$, since Σ is full with respect to U.

Now $|\Sigma| = |\Omega\setminus\Sigma|$, as Σ is a moiety, so there is an automorphism $x \in E$ of order 2 such that $M^\Sigma x = M^{\Omega\setminus\Sigma}$ and $M^{\Omega\setminus\Sigma} x = M^\Sigma$. Then $\pi_\Sigma f \pi_\Sigma$, $\pi_\Sigma f \pi_{\Omega\setminus\Sigma} x$, $x\pi_{\Omega\setminus\Sigma} f \pi_\Sigma$, $x\pi_{\Omega\setminus\Sigma} f \pi_{\Omega\setminus\Sigma} x \in E_{[\Sigma]}$. Hence, $\pi_\Sigma f \pi_{\Omega\setminus\Sigma} = \pi_\Sigma f \pi_{\Omega\setminus\Sigma} x^2 \in E_{[\Sigma]} x$, $\pi_{\Omega\setminus\Sigma} f \pi_\Sigma = x^2 \pi_{\Omega\setminus\Sigma} f \pi_\Sigma \in xE_{[\Sigma]}$, and $\pi_{\Omega\setminus\Sigma} f \pi_{\Omega\setminus\Sigma} = x^2 \pi_{\Omega\setminus\Sigma} f \pi_{\Omega\setminus\Sigma} x^2 \in xE_{[\Sigma]} x$. So $f \in E_{[\Sigma]} + E_{[\Sigma]} x + xE_{[\Sigma]} + xE_{[\Sigma]} x = \pi_\Sigma U \pi_\Sigma + \pi_\Sigma U \pi_\Sigma x + x\pi_\Sigma U \pi_\Sigma + x\pi_\Sigma U \pi_\Sigma x$. □

The proof of the following lemma is set-theoretic in nature, so aside from a few minor adjustments, we present it here the way it appears in [1].

Lemma 2. *Let $(U_i)_{i\in I}$ be any family of subsets of E such that $\bigcup_{i\in I} U_i = E$ and $|I| \leq |\Omega|$. Then Ω contains a full moiety with respect to some U_i.*

Proof. Since $|\Omega|$ is infinite and $|I| \leq |\Omega|$, we can write Ω as a union of disjoint moieties Σ_i, $i \in I$. Suppose that there are no full moieties with respect to U_i for any $i \in I$. Then in particular, Σ_i is not full with respect to U_i for any $i \in I$. So for every $i \in I$ there exists an endomorphism $f_i \in \mathrm{End}_R(M^{\Sigma_i})$ which is not the restriction to M^{Σ_i} of any member of $(U_i)_{\{\Sigma_i\}}$. Now if we take $f \in E$ to be the endomorphism whose restriction to each M^{Σ_i} is f_i, then f is not in U_i for any $i \in I$, contradicting $\bigcup_{i\in I} U_i = E$. □

3 Generating Sets

We are now ready to prove our main results.

Theorem 3. *Let R be a ring, M a nonzero left R-module, Ω an infinite set, and $E = \mathrm{End}_R(\bigoplus_{i\in\Omega} M)$ (or $\mathrm{End}_R(\prod_{i\in\Omega} M)$). Suppose that $(R_i)_{i\in I}$ is a chain of subrings of E such that $\bigcup_{i\in I} R_i = E$ and $|I| \leq |\Omega|$. Then $E = R_i$ for some $i \in I$.*

Proof. By the preceding lemma, Ω contains a full moiety with respect to some R_i. Thus, Lemma 1 implies that $E = \langle R_i \cup \{x, y\}\rangle$ for some $x, y \in E$. But, by the hypotheses on $(R_i)_{i\in I}$, $R_i \cup \{x, y\} \subseteq R_j$ for some $j \in I$, and hence $E = \langle R_i \cup \{x, y\}\rangle \subseteq R_j$, since R_j is a subring. □

Definition 4. *Let R be a ring and U a subset of R. We will say that $r \in R$ is represented by a* ring word of length 1 *in elements of U if $r \in U \cup \{0, 1, -1\}$, and, recursively, that $r \in R$ is represented by a* ring word of length n *in elements of U if $r = p + q$ or $r = pq$ for some elements $p, q \in R$ which can be represented by ring words of lengths m_1 and m_2 respectively, with $n = m_1 + m_2$.*

Theorem 5. *Let R be a ring, M a nonzero left R-module, Ω an infinite set, and $E = \mathrm{End}_R(\bigoplus_{i\in\Omega} M)$ (or $\mathrm{End}_R(\prod_{i\in\Omega} M)$). If U is a generating set for E as a ring, then there exists a positive integer n such that every element of E is represented by a ring word of length at most n in elements of U.*

Proof. For $i = 1, 2, 3, \ldots$, let U_i be the set of elements of R that can be expressed as ring words in elements of U of length $\leq i$. Then $\bigcup_{i=1}^{\infty} U_i = E$, since U is a generating set. Since $\aleph_0 \leq |\Omega|$, Lemma 2 implies that Ω contains a full moiety with respect to some U_i. By Lemma 1, there exist $x, y \in E$ such that $E = yU_iy + yU_iyx + xyU_iy + xyU_iyx$. Let $k \geq i$ be such that U_k contains $U_i \cup \{x, y\}$. We then have $E = U_k^3 + U_k^4 + U_k^4 + U_k^5$.

Now, for any positive integer m, U_k^m consists of ring words of length $\leq mk$ in elements of U. Thus $E = U_k^3 + U_k^4 + U_k^4 + U_k^5$ consists of ring words of length $\leq 16k$ in elements of U. $\square$

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References

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